Optimal Laminate Configurations of Cylindrical Shells for Axial Buckling

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Optimum laminate configurations for laminated cylindrical shells under axial compression are investigated and obtained under buckling constraints. Complete freedom is given in the selection of the ply angle variation through the thickness. Twelve lamination parameters are introduced to describe laminate properties, and the optimal values of these parameters are obtained numerically. It is shown that there are many different optimal configurations which give the same buckling load. The optimality condition for the laminate configurations is derived semiempirically from the numerical results in terms of the lamination parameters. The maximum buckling load is obtained in terms of material properties. Some examples of optimal laminate configurations are presented. It is shown that one of the optimal laminate configurations can be obtained when an infinite number of infinitely thin layers are arranged so that the shell becomes quasi-isotropic in the shell surface and quasihomogeneous through the thickness.

Nomenclature

A_{ij},B_{ij},D_{ij} = nondimensional stiffnesses, see Eqs. (3) and (4)
C_i = defined in Eq. (13)
E = Young's modulus
$\tilde{E}_T,\tilde{G}_{LT}$ = normalized transverse Young's modulus and shear modulus, respectively, see Eq. (6)
F_1,F_2 = defined in Eq. (28)
$f_{\max}(g,\phi)$, $f_{\min}(g,\phi)$ = maximum and minimum value, respectively, of the function g with respect to its argument ϕ
G = shear modulus
K = nondimensional buckling load coefficient, see Eq. (17)
M_x,M_y,M_{xy} = resultant moment per unit width
m,n = numbers of axial half-waves and circumferential waves, respectively
N = applied axial compressive load per unit width
N_{cr} = value of N which causes instability
N_x,N_y,N_{xy} = resultant stresses
p,q = defined in Eq. (12)
$q_{\min}$ = defined in Eq. (15)
R,t,L = radius, thickness, and length of cylindrical shells, respectively
u,v,w = displacements in the x, y, and z directions, respectively
W_i = defined in Eq. (5)
x,y,z = axial, circumferential, and radial coordinates, respectively
$\epsilon_x,\epsilon_y,\gamma_{xy}$ = strains
ζ_i = lamination parameter, see Eq. (7)
θ = angle between x axis and fiber or principal axis of layer
$\kappa_x,\kappa_y,\kappa_{xy}$ = changes of curvature
ν = Poisson's ratio
ξ = defined by Eq. (8)

Superscripts

0 = associated with midsurface

Subscripts

L = longitudinal
opt = optimal
T = transversal

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Introduction

MANY papers[1-5] on the stability analysis of laminated composite cylindrical shells have shown that the laminate configurations have great influence on the buckling loads and, consequently, that improvement of the strength by means of tailoring is possible or even necessary. Many efforts to identify the optimal laminate configuration and some limited results have been reported. Tasi[1] has analyzed the stability of cylindrical shells composed of three layers (two layers with fibers oriented circumferentially and one layer with fibers oriented axially) under axial compressive load and internal pressure. Hirano[2] has investigated the buckling load of angle-ply laminated cylindrical shells under axial compression and has obtained the optimal lamination angle under the assumption that the lamination angle is constant through the thickness. Kobayashi et al.[3] have obtained the optimal stacking sequence of layers with 0 deg, 90 deg, $+\theta$, and $-\theta$ fiber directions and the optimal value of θ to obtain a maximum buckling load of two-, three-, four-, six-, and eight-layer composite cylinders, respectively. He also investigated the effect of prebuckling deformation on buckling loads. Recently, Nshanian and Pappas[5] have applied a special mathematical programming (MP) algorithm to determine the optimal ply angle variation through the thickness under the restriction of symmetric, orthotropic, angle-ply laminate configurations, considering the minimum natural frequency or the axial and internal/external pressure buckling load. They approximated the ply angle distribution by means of continuous piecewise-linear functions or discontinuous piecewise-constant functions, and obtained some nearly optimal laminate configurations.

However, a general demonstration of the optimal laminate configurations has not yet been shown, since in these previous papers the optimization was carried out under many restrictive conditions. It may not be necessary for structural designers to optimize structures strictly, but it is very important for them to be aware of the optimal configuration as a general guideline. In this paper, the optimal laminate configuration is obtained by applying the energy method, based on the Donnell-type shell theory, to the axial buckling of thin, not very short, simply supported laminated composite cylindrical shells. Complete freedom is given for the first time in the selection of the ply angle distribution through the thickness. Twelve lamination parameters, which are functionals of the distribution function of the ply angles through the shell thickness, are introduced, and it is shown that the mechanical characteristics of the shells are determined uniquely by these 12 parameters for a given material. The optimal values of these parameters, which maximize the buckling load for the given materials and dimensional parameters, and the corresponding buckling load coefficient, are calculated. The condition of the optimal laminate configuration is derived semiempirically in terms of the lamination parameters. Since the objective of this paper is to identify the optimal laminate configuration, i.e., to investigate the relationship between the change of buckling load and the change in the parameter values, relatively simple buckling anaysis is used under the assumption that the shells are thin and not very short. The thickness of the shell is assumed to be uniform and the material of each layer is assumed to be the same.

Buckling Analysis

The shells are assumed to be composed of anisotropic layers whose material properties are identical. The ply angle θ is considered to be an arbitrary function of z defined in the region $-t/2<z<t/2$. The strains at z are assumed to be related to the midsurface strains and the curvature changes as follows.

$$\epsilon_x=\epsilon_x^0+z\kappa_x,\quad \epsilon_y=\epsilon_y^0+z\kappa_y,\quad \gamma_{xy}=\gamma_{xy}^0+z\kappa_{xy} \tag{1}$$

The stress and the moment resultants are defined as follows by assuming that the shells are very thin:

$$(N_x,N_y,N_{xy},M_x,M_y,M_{xy})=\int_{-t/2}^{t/2}(\sigma_x,\sigma_y,\tau_{xy},z\sigma_x,z\sigma_y,z\tau_{xy})\mathrm{d}z \tag{2}$$

Then, the following relations between the stress and moment resultants and the midsurface strains and curvatures can be derived from the preceding equations, together with the classical lamination theory.[6]

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = E_L t \begin{bmatrix} A_{11} & A_{12} & A_{16} & tB_{11} & tB_{12} & tB_{16} \\ A_{12} & A_{22} & A_{26} & tB_{12} & tB_{22} & tB_{26} \\ A_{16} & A_{26} & A_{66} & tB_{16} & tB_{26} & tB_{66} \\ tB_{11} & tB_{12} & tB_{16} & t^2D_{11} & t^2D_{12} & t^2D_{16} \\ tB_{12} & tB_{22} & tB_{26} & t^2D_{12} & t^2D_{22} & t^2D_{26} \\ tB_{16} & tB_{26} & tB_{66} & t^2D_{16} & t^2D_{26} & t^2D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{3}$$

where

$$A_{11}=W_1+W_2\zeta_1+W_3\zeta_3,\qquad A_{12}=W_4-W_3\zeta_3,$$
$$A_{22}=W_1-W_2\zeta_1+W_3\zeta_3,\qquad A_{66}=W_5-W_3\zeta_3,$$
$$A_{16}=-W_2\zeta_2/2-W_3\zeta_4,\qquad A_{26}=-W_2\zeta_2/2+W_3\zeta_4$$
$$B_{11}=(W_2\zeta_5+W_3\zeta_7)/4,\qquad B_{12}=-W_3\zeta_7/4$$
$$B_{22}=(-W_2\zeta_5+W_3\zeta_7)/4,\qquad B_{66}=-W_3\zeta_7/4$$
$$B_{16}=(-W_2\zeta_6/2-W_3\zeta_8)/4,\quad B_{26}=(-W_2\zeta_6/2+W_3\zeta_8)/4$$

$$D_{11}=(W_1+W_2\zeta_9+W_3\zeta_{11})/12,\quad D_{12}=(W_4-W_3\zeta_{11})/12$$
$$D_{22}=(W_1-W_2\zeta_9+W_3\zeta_{11})/12,\quad D_{66}=(W_5-W_3\zeta_{11})/12$$
$$D_{16}=(-W_2\zeta_{10}/2-W_3\zeta_{12})/12,$$
$$D_{26}=(-W_2\zeta_{10}/2+W_3\zeta_{12})/12 \tag{4}$$

$$W_1=(1/4)(1+\tilde{E}_T+2\nu_T)/(1-\nu_L\nu_T)+\tilde{G}_{LT}$$
$$W_2=(1/2)(1-\tilde{E}_T)/(1-\nu_L\nu_T)$$
$$W_3=(1/4)(1+\tilde{E}_T-2\nu_T)/(1-\nu_L\nu_T)-\tilde{G}_{LT}$$
$$W_4=(1/4)(1+\tilde{E}_T+2\nu_T)/(1-\nu_L\nu_T)-\tilde{G}_{LT}$$
$$W_5=(1/4)(1+\tilde{E}_T-2\nu_T)/(1-\nu_L\nu_T) \tag{5}$$

$$\tilde{E}_T=E_T/E_L,\quad \tilde{G}_{LT}=G_{LT}/E_L \tag{6}$$

$$\zeta_1=\frac{1}{2}\int_{-1}^{1}\cos 2\theta\,\mathrm{d}\xi,\qquad \zeta_2=\frac{1}{2}\int_{-1}^{1}\sin 2\theta\,\mathrm{d}\xi$$
$$\zeta_3=\frac{1}{2}\int_{-1}^{1}\cos^2 2\theta\,\mathrm{d}\xi,\qquad \zeta_4=\frac{1}{2}\int_{-1}^{1}\cos 2\theta\sin 2\theta\,\mathrm{d}\xi$$
$$\zeta_5=\int_{-1}^{1}\xi\cos 2\theta\,\mathrm{d}\xi,\qquad \zeta_6=\int_{-1}^{1}\xi\sin 2\theta\,\mathrm{d}\xi$$
$$\zeta_7=\int_{-1}^{1}\xi\cos^2 2\theta\,\mathrm{d}\xi,\qquad \zeta_8=\int_{-1}^{1}\xi\cos 2\theta\sin 2\theta\,\mathrm{d}\xi$$
$$\zeta_9=\frac{3}{2}\int_{-1}^{1}\xi^2\cos 2\theta\,\mathrm{d}\xi,\qquad \zeta_{10}=\frac{3}{2}\int_{-1}^{1}\xi^2\sin 2\theta\,\mathrm{d}\xi$$
$$\zeta_{11}=\frac{3}{2}\int_{-1}^{1}\xi^2\cos^2 2\theta\,\mathrm{d}\xi,\quad \zeta_{12}=\frac{3}{2}\int_{-1}^{1}\xi^2\cos 2\theta\sin 2\theta\,\mathrm{d}\xi \tag{7}$$

$$\xi=2z/t \tag{8}$$

Equations (3), (4), and (7) show that the characteristics of the laminated composite shells are defined by only 12 laminate parameters for any given material characteristics of the layer. To the author's knowledge, some of these parameters or their equivalents were first introduced by Miki[7] and Fukunaga and Hirano.[8]

In this paper, the axial compressive buckling load is estimated by the Rayleigh-Ritz method. The Donnell-type strain-displacement relations are used in the analysis. It is assumed that the cylindrical shells are not so long as to allow Euler buckling to occur first. The following buckling deformation functions are used, which satisfy the kinematic boundary conditions ($w=0$, $v=0$) of simple support ($S-2$).

$$w=w_{mn}\sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{ny}{R}\right),\quad u=u_{mn}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{ny}{R}\right)$$
$$v=v_{mn}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{ny}{R}\right) \tag{9}$$

It should be noted that these displacements satisfy both of the Donnell-type equilibrium equations and the simply supported boundary conditions ($S-2$, i.e, $w=0$, $M_x=0$, $N_x=0$, $v=0$) rigorously, only if

$$A_{16}=A_{26}=B_{16}=B_{26}=D_{16}=D_{26}=0 \tag{10}$$

If these values are not zero, neither the equilibrium equations nor the natural boundary conditions ($M_x = 0$, $N_x = 0$) are satisfied and, therefore, overestimated approximate eigenvalues will be obtained. The buckling load is obtained as the lowest eigenvalue of the equations of stationary conditions of the total potential energy. The eigenvalues are obtained as a function of n and m or the parameters p and q as follows:

$$N(p,q) = (E_L t^2/R)\{q(t/R)C_1(p) + C_2(p) + (R/t)C_3(p)/q\} \quad (11)$$

where

$$q = \bar{m}^2 = (m\pi R/L)^2, \quad p = (n/\bar{m})^2 \quad (12)$$

$$C_1(p) = C_5(p) - C_6(p)/C_4(p)$$

$$C_2(p) = 2B_{12} + C_7(p)/C_4(p)$$

$$C_3(p) = C_8/C_4(p)$$

$$C_4(p) = A_{11}A_{66} - (A_{12}^2 + 2A_{12}A_{66} - A_{11}A_{22})p + A_{22}A_{66}p^2$$

$$C_5(p) = D_{11} + 2(D_{12} + 2D_{66})p + D_{22}p^2$$

$$C_6(p) = A_{66}B_{22}^2p^4 + (A_{11}B_{22}^2 + A_{22}B_6^2 - 2A_{12}B_6B_{22})p^3 + 2(A_{11}B_6B_{22} + A_{22}B_6B_{11} - A_{12}B_{11}B_{22} - A_{12}B_6^2 - A_{66}B_{11}B_{22})p^2 + (A_{11}B_6^2 + A_{22}B_{11}^2 - 2A_{12}B_{11}B_6)p + A_{66}B_{11}^2$$

$$C_7(p) = 2\{-A_{12}A_{66}B_{22}p^2 + (A_{11}A_{66}B_{22} + A_{12}^2B_6 + A_{66}A_{22}B_{11} - A_{11}A_{22}B_6)p - A_{12}A_{66}B_{11}\}$$

$$C_8 = A_{66}(A_{11}A_{22} - A_{12}^2) \quad (13)$$

$$B_6 = B_{12} + 2B_{66} \quad (14)$$

It should be noted that the parameters ζ_2, ζ_4, ζ_6, ζ_8, ζ_{10}, and ζ_{12} have disappeared from the preceding equations. Equation (11) is the same as the exact eigenvalue obtained directly from the Donnell-type equilibrium equations, when the values of A_{16}, A_{26}, B_{16}, B_{26}, D_{16}, and D_{26} are all zero.

The critical buckling load is the minimum value of N with respect to the integers n and m. However, in this report, for the sake of simplicity, the variables p and q, which are the functions of n and m, are considered to be continuous. When the values of C_1 and C_3 are positive, the value of q, which minimizes N, is

$$q_{\min} = (R/t)\sqrt{C_2(p)/C_1(p)} \quad (15)$$

and the corresponding minimum value of N becomes

$$N(p,q_{\min}) = (E_L t^2/R)\{2\sqrt{C_1(p)C_3(p)} + C_2(p)\} \quad (16)$$

The axial compressive load per unit width which causes instability, N_{cr}, is the minimum value of $N(p,q_{\min})$ with respect to p ($0 \leq p$), and the nondimensional buckling load coefficient is introduced as follows:

$$K(\zeta_1,\zeta_3,\zeta_5,\zeta_7,\zeta_9,\zeta_{11}) = N_{cr}R/(E_L t^2) = f_{\min}\{N(p,q_{\min}),\ p\}R/(E_L t^2) = f_{\min}\{2\sqrt{C_1(p)C_3(p)} + C_2(p),\ p\} \quad (17)$$

where the function $f_{\min}(g,\phi)$ denotes the minimum value of the function g with respect to its argument ϕ. When the value of p increases infinitely, as can be seen from Eqs. (13) and (15), $q_{\min}$ becomes proportional to p^{-2} as a limit. Therefore, both values of $\bar{m} = \sqrt{q} \propto p^{-1}$ and $n = \sqrt{pq} \propto p^{-1/2}$ become zero as a limit. This is an unrealistic situation. In spite of this, the minimum value of $N(p,q_{\min})$ is searched throughout the full range of $0 \leq p$ in this paper. The above treatment is not suitable for very short cylindrical shells.

Optimal Values of ζ_i (i=2, 4, 6, 8, 10, 12)

As mentioned previously, the assumed buckling deformation functions, Eq. (9), satisfy the Donnell-type equations of equilibrium and the natural boundary conditions of simple support only if Eq. (10) is satisfied, i.e., only if the shear-extension couplings are negligible. Equation (10) is equivalent to the following equations when $W_2W_3 \neq 0$, i.e., when the ply is anisotropic.

$$\zeta_2 = \zeta_4 = \zeta_6 = \zeta_8 = \zeta_{10} = \zeta_{12} = 0 \quad (18)$$

These facts mean that Eq. (9) cannot express the true buckling deformation if Eq. (18) is not satisfied. Since the Rayleigh-Ritz method gives overestimated approximations for eigenvalues whenever the assumed deformation functions cannot express the true deformation, the true eigenvalues, which can be obtained by using true buckling deformation functions, are less than the approximate values given by Eq. (11) or (17) if any values of ζ_i (i=2, 4, 6, 8, 10, 12) are not zero. Furthermore, as has been noted, Eqs. (11) and (17) are independent of the values of ζ_i (i=2, 4, 6, 8, 10, 12) explicitly as well as implicitly. Therefore, it is concluded that the optimal values of ζ_i (i=2, 4, 6, 8, 10, 12) which maximize the true eigenvalues are given by Eq. (18), because the true eigenvalues become identical to Eq. (11) or (17) only if Eq. (18) is satisfied, and otherwise, the true values are less than Eq. (11) or (17). This conclusion is valid not only for relatively long cylindrical shells subjected to the axial compression, but also for very short shells subjected to axial compression as well as cylindrical shells subjected to the combined axial and internal/external pressure loads.

The above-mentioned optimality conditions for ζ_i (i=2, 4, 6, 8, 10, 12) qualitatively coincide with, for example, the results of Uemura and Kasuya,[4] who have shown, by some numerical examples, that the axial buckling loads usually decrease when the values of A_{16}, A_{26}, B_{16}, B_{26}, D_{16}, and D_{26} become nonnegligible. However, to the author's knowledge, to date the above optimality conditions have not been pointed out as clearly as in the present paper.

Optimization with Respect to ζ_i (i=1, 3, 5, 7, 9, 11)

In this report, the values of the parameter ζ_i (i=1, 3, 5, 7, 9, 11) which make the value of K maximum are numerically obtained by the following method.

$$K_{\text{opt}} = f_{\max}(f_{\max}(f_{\max}(f_{\max}(f_{\max}(f_{\max}(K,\zeta_1),\zeta_3),\zeta_5),\zeta_7),\zeta_9),\zeta_{11}) \quad (19)$$

where the function $f_{\max}(g,\phi)$ denotes the maximum value of the function g with respect to its argument ϕ. Each function $f_{\max}$ is calculated numerically by direct search of the maximum value. For each evaluation of the value of $f_{\max}(f_{\max}(f_{\max}(f_{\max}(f_{\max}(K,\zeta_1),\zeta_3),\zeta_5),\zeta_7),\zeta_9)$ for certain values of ζ_{11} (necessary in the process of the optimization with respect to ζ_{11}), optimization with respect to ζ_9 is required. An optimization with respect to ζ_7 is carried out for each evaluation of the value of $f_{\max}(f_{\max}(f_{\max}(f_{\max}(K,\zeta_1),\zeta_3),\zeta_5),\zeta_7)$ for certain values of ζ_9 that is required in the process of optimization with respect to ζ_9, and so on. Therefore, a great deal of computational time is required, especially for problems with a large number of design parameters. However, this method is utilized in order to obtain reliable results, because the maximum point can be successfully obtained by this method even if the surface of function K has a very sharp ridge, as in the present cases.

It can be seen from Eq. (7) that the values of ζ must satisfy certain conditions to assure the existence of the real function $\theta(z)$. The possible range of the value of ζ_i is, for example, a function of the values of ζ_j $(j=1,2,\ldots,i-1)$. Therefore, the optimal values of ζ must be found under the above constraints. In this paper, however, the optimal values of ζ are searched first without such constraints and, subsequently, are shown to satisfy these restrictive conditions by ascertaining if there exists any real function $\theta(z)$ corresponding to them.

Results of Numerical Calculation

A large number of numerical calculations for optimization are carried out with various initial values. Random values of ζ which satisfy the following evident restrictive conditions are used as the initial values without any investigation of the existence of any corresponding ply angle distribution function $\theta(z)$.

$$-1\leq\zeta_i\leq 1 \quad (i=1,5,7,9); \quad 0\leq\zeta_i\leq 1 \quad (i=3,11) \tag{20}$$

$$\zeta_1^2\leq\zeta_3, \quad \zeta_9^2\leq\zeta_{11} \tag{21}$$

$$C_1(p)>0 \quad (0\leq p); \quad C_3(p)>0 \quad (0\leq p) \tag{22}$$

The conditions in Eq. (21) can be derived from Schwarz's inequality.[7] The conditions in Eq. (22) are introduced, because otherwise [as can be seen from Eq. (11)] the value of N can become negative or zero, which are unrealistic values.

It is found from the numerical results that the optimal point cannot be determined uniquely. That is, there are many sets of the values of ζ which give the same optimal value of K. However, if one of the values of ζ_3, ζ_7, and ζ_{11} is set to a specified value, the corresponding optimal point can be determined uniquely, regardless of the initial values and the specified values. The optimal point gives the same optimal value of K as the one obtained without the above restriction on the value of ζ_3, ζ_7, or ζ_{11}. This fact indicates that the optimal value obtained is not only a locally maximal value.

Furthermore, the following points have been deduced from the above-mentioned numerical results with very high accuracy for all of the different anisotropic material properties investigated here.

1) The optimal values of ζ_1, ζ_5, and ζ_9 are always given by

$$\zeta_1=\zeta_5=\zeta_9=0 \tag{23}$$

2) The value of $N(p,q_{\min})$ does not depend upon the value of p whenever the values of ζ are optimal.

3) One of the optimal sets of the values of ζ is given by

$$\zeta_1=\zeta_5=\zeta_7=\zeta_9=0, \quad \zeta_3=\zeta_{11}=½ \tag{24}$$

A few examples of numerical results for different initial values are listed in Table 1, which demonstrate points 1 and 3 above. The example of numerical results listed in Table 2 demonstrates point 2. In all of the cases listed in the tables, the value of ζ_{11} is fixed to the specified values. As can be seen in the tables, the above-mentioned points are indicated by the results of this calculation with a very high numerical accuracy. The numerical accuracy has been increased by the stricter termination condition in the iteration process of the numerical calculation and by the increased allocation of bits per variable in computer computation, and it is likely that even more accuracy can be expected if desired.

Optimality Conditions

As has been mentioned, the results of the numerical calculations show that Eq. (24) gives one of the optimal points. After the substitution of Eq. (24) into Eq. (4), the following optimal reduced buckling load coefficients can be derived from Eqs. (13) and (17), where the value of $N(p,q_{\min})$ becomes independent of p.

$$K_{\text{opt}}=[(1+\tilde{E}_T+2\nu_T)\{1+\tilde{E}_T-2\nu_T+4\tilde{G}_{LT} \times(1-\nu_T^2/\tilde{E}_T)\}/24]^{½}/(1-\nu_T^2/\tilde{E}_T) \tag{25}$$

This equation gives the reduced buckling load coefficient for the optimal laminate configuration, which is the same as the classical buckling load coefficient of quasi-isotropic homogeneous shells. This equation gives the same value of K as the optimal values directly obtained by the numerical calculation for all of the cases investigated herein.

From the numerical calculation result of point 2 of the preceding section, the following equation, for example, can be derived for all optimal sets of the values of ζ:

$$N(p=0,\ q_{\min})=N(p=1,\ q_{\min})=\text{Eq. (25)}\times E_L t^2/R \tag{26}$$

Using the numerical result, point 1 of the preceding section, and the above equation, the following relation can be derived between the optimal values of ζ_3, ζ_7, and ζ_{11} from Eqs. (4), (5), (13), (14), (16), and (25):

$$(\zeta_3-0.5)=F_1(\zeta_{11}-0.5), \quad \zeta_7=F_2(\zeta_{11}-0.5) \tag{27}$$

where

$$F_1=(W_1+W_3-W_4)/(W_1+W_4)$$
$$=\{1+\tilde{E}_T-2\nu_T+4\tilde{G}_{LT}(1-\nu_T^2/\tilde{E}_T)\}/\{2(1+\tilde{E}_T+2\nu_T)\}$$
$$F_2=2\sqrt{F_1/3} \tag{28}$$

The optimal values of ζ_3 and ζ_7 obtained from Eq. (27) for the given value of ζ_{11} are listed in Table 1 and compared with the values obtained by direct numerical optimization. We can see that these values coincide with very high accuracy. A similar situation is observed for all of the different materials investigated here, although they are not listed. This fact strongly suggests that points 1-3 of the preceding section, which are deduced from the numerical calculation results, are true. As a result, it can be concluded that the optimality conditions for laminate configurations are given by Eq. (27) together with the following equation:

$$\zeta_1=\zeta_2=\zeta_4=\zeta_5=\zeta_6=\zeta_8=\zeta_9=\zeta_{10}=\zeta_{12}=0 \tag{29}$$

It is empirically known that the anisotropy of the laminated cylindrical shells usually decreases the axial buckling loads. This point coincides with the above results. However, the above conclusion shows that there also exist anisotropic optimal configurations which give the same buckling loads as the isotropic case.

Existence of the Ply Angle Distribution Function $\theta(z)$

If we consider a lamination composed of equally and alternately stacked thin layers with the angles of $\pi/6$, 0, and $-\pi/6$, it can be seen that the values of ζ of this lamination become

$$\zeta_1=\zeta_2=\zeta_4=\zeta_5=\zeta_6=\zeta_7=\zeta_8=\zeta_9=\zeta_{10}=\zeta_{12}=0, \quad \zeta_3=\zeta_9=0.5 \tag{30}$$

Table 1 Some examples of initial values of ζ, final optimal values of ζ and K, and the values of ζ obtained with Eq. (27) ($\tilde{E}_T = 0.06$, $\nu_T = 0.02$, $\tilde{G}_{LT} = 0.035$)

	Initial	Final	Eq. (27)	Initial	Final	Eq. (27)
ζ_1	0.1542	−0.00000	—	0.3483	0.00000	—
ζ_3	0.0535	0.60537	0.60537	0.9015	0.55269	0.55268
ζ_5	0.0481	0.00000	—	−0.5276	−0.00000	—
ζ_7	0.3077	0.16763	0.16763	0.0898	0.08382	0.08381
ζ_9	−0.2317	−0.00000	—	0.2060	0.00000	—
ζ_{11}	0.7000	0.70000	(0.7)	0.6000	0.60000	(0.6)
K		0.232033			0.232033	
ζ_1	0.2561	0.00000	—	−0.0873	0.00000	—
ζ_3	0.5820	0.50000	0.50000	0.5213	0.44732	0.44732
ζ_5	0.1474	−0.00000	—	−0.0759	0.00000	—
ζ_7	−0.6611	−0.00000	0.00000	0.4145	−0.08381	−0.08381
ζ_9	0.4525	0.00000	—	−0.3885	0.00000	—
ζ_{11}	0.5000	0.50000	(0.5)	0.4000	0.40000	(0.4)
K		0.232033			0.232033	

Table 2 An example of initial and final values of $N(p,q_{\min})R/(E_L t^2)$ and ζ ($\tilde{E}_T = 0.06$, $\nu_T = 0.02$, $\tilde{G}_{LT} = 0.035$)

	p	Initial	Final
$\dfrac{N(p,q_{\min})R}{E_L t^2}$	0	0.3214473	0.2320331
	0.01	0.2907025	0.2320331
	0.1	0.1461927	0.2320331
	1.0	0.0314076	0.2320331
	10.0	0.0901389	0.2320331
	100.0	0.2261810	0.2320331
	∞	0.2622472	0.2320331
ζ_1		0.1578	−0.0000002
ζ_3		0.8998	0.3946335
ζ_5		−0.4780	−0.0000017
ζ_7		−0.6711	−0.1676238
ζ_9		0.3779	−0.0000011
ζ_{11}		0.3000	0.3000000

when the number of layers increases infinitely and the thickness of each layer decreases infinitely. Generally, Eq. (30) corresponds to the laminated shells with the infinite number of infinitely thin layers arranged so that the shell becomes quasi-isotropic in the x-y plane and quasihomogeneous across the thickness. These specific optimal configurations have the advantage of ease of analysis since we know much about isotropic homogeneous shells.

Let us consider the following laminate configurations: For example, a configuration with part of the thickness $\eta < \xi < 1$ composed of equally and alternately stacked infinitely thin layers with the angles of $+\theta_{11}$, $-\theta_{11}$, $+\theta_{12}$, and $-\theta_{12}$, and the rest of the thickness $-1 < \xi < \eta$ composed of equally and alternately stacked infinitely thin layers with the angles of $+\theta_{21}$, $-\theta_{21}$, $+\theta_{22}$, and $-\theta_{22}$, where

$$\cos 2\theta_{11} = \sqrt{0.5+\alpha_1}, \quad \cos 2\theta_{12} = -\sqrt{0.5+\alpha_1}$$

$$\cos 2\theta_{21} = \sqrt{0.5+\alpha_2}, \quad \cos 2\theta_{22} = -\sqrt{0.5+\alpha_2} \tag{31}$$

When η, α_1, and α_2 satisfy the following relations, it can be seen that the values of ζ of this laminated shell satisfy Eqs. (27) and (29).

$$\frac{1}{2}\left\{\int_\eta^1 \left(\frac{1}{2}+\alpha_1\right)\mathrm{d}\xi + \int_{-1}^\eta \left(\frac{1}{2}+\alpha_2\right)\mathrm{d}\xi\right\} = \frac{1}{2} + F_1\delta$$

$$\int_\eta^1 \left(\frac{1}{2}+\alpha_1\right)\xi\mathrm{d}\xi + \int_{-1}^\eta \left(\frac{1}{2}+\alpha_2\right)\xi\mathrm{d}\xi = F_2\delta$$

$$\frac{3}{2}\left\{\int_\eta^1 \left(\frac{1}{2}+\alpha_1\right)\xi^2\mathrm{d}\xi + \int_{-1}^\eta \left(\frac{1}{2}+\alpha_2\right)\xi^2\mathrm{d}\xi\right\} = \frac{1}{2} + \delta \tag{32}$$

$$-0.5 < \alpha_1 < 0.5, \quad -0.5 < \alpha_2 < 0.5 \tag{33}$$

where δ is an arbitrary constant.

Equation (32) corresponds to Eq. (27). The eigenvalue η and the elements of eigenvector α_1 and α_2 of Eq. (32) are found as follows, under the condition of $-1 < \eta < 1$.

$$\eta = (1-F_1)/F_2 \tag{34}$$

$$\alpha_2 = \alpha_1\{4(1-F_2) - 3(1-F_1)^2\}/\{4(1+F_2) - 3(1-F_1)^2\} \tag{35}$$

where α_1 is an arbitrary constant under the condition Eq. (33).

The above equations show examples of optimal laminate configurations giving a freedom of the value of α_1. When α_1 and α_2 are not zero, the above examples of laminate configurations are not only anisotropic but also unsymmetric, having a nonzero value for ζ_7. In other words, B_{11}, B_{12}, B_{22}, and B_{66} are not zero.

Since some examples of the real ply angle distribution functions $\theta(z)$ corresponding to the optimal values of ζ have been shown to exist, it can be concluded that the optimal sets of the values of ζ obtained in previous sections (without any rigorous restrictions) are real ones.

Conclusions

The optimal laminate configurations for composite cylindrical shells under axial compression are obtained under buckling constraints. For the first time, complete freedom is given in the selection of the fiber angle distribution through the thickness. It is shown clearly that the shear-extension couplings should be negligible for the optimal laminate configurations. Furthermore, the numerical results and the investigation based thereon show that the condition necessary for an optimal laminate configuration is given by Eqs. (27) and (29), and that the maximized buckling load coefficient is given by Eq. (25). It is also found that there are many optimal laminate configurations, both symmetric and asymmetric. One of the optimal configurations is shown to be the lamination with an infinite number of infinitely thin layers arranged so that the shell becomes quasi-isotropic in the shell surface and quasihomogeneous through the thickness.

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